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A note on the field-theoretical description of superfluids

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ABSTRACT

Recently, a Lagrangian description of superfluids attracted some interest from the fluid/gravity-correspondence viewpoint. In this respect, the work of Dubovksy et al. has proposed a new field theoretical description of fluids, which has several interesting aspects. On another side, we have recently provided a supersymmetric extension of the original works. In the analysis of the Lagrangian structures a new invariant appeared which, although related to known invariants, provides, in our opinion, a better parametrization of the fluid dynamics in order to describe the fluid/superfluid phases.

Motivated by fluid/gravity correspondence [2–5] and the recent developments in the field theory description of fluids by [6–10], in [1] we have given a field-theoretical description of fluid dynamics suitably extended to a supersymmetric framework. The supersymmetric formulation was achieved by suitable extension in superspace of the description of a relativistic non-dissipative fluid in terms of an effective Lagrangian whose fields are identified with the comoving-coordinates $\phi^I(x)$ ($I = 1, \dots, d$ labeling the spatial indices), as first considered in [6].[1]

In Refs. [7–9] the authors also introduced a further field ψ representing a U(1)-phase related to the presence of a charge. Assuming that the Lagrangian is invariant under a set of spatial symmetries and under an internal gauge symmetry acting on ψ named *chemical shift symmetry*, they were able to determine two fundamental invariants, which from the thermodynamical point of view turn out to be the entropy density s and the chemical potential μ.

The extension to a supersymmetric framework was naturally obtained in Ref. [1] by promoting the fields to superfields so that each field has a fermionic partner. In particular the supersymmetrization requires the introduction, besides the fermionic coordinates $\theta^\alpha(x)$[2] partners of the $\phi^I(x)$, an additional fermionic field $\tau(x)$ partner of the local coordinate $\psi(x)$ also transforming under the chemical-shift symmetry. In that way, the basic ingredients of the Lagrangian description are a set of superfields, invariant under the same set of symmetries of the bosonic theories, their bosonic part coinciding with the invariants discussed in the quoted papers, namely the entropy and the chemical potential.

While in Ref. [1] we were mainly focusing on the supersymmetric extension of the bosonic entropy current, we also discussed the possibility of considering further new Poincaré invariants Z^I out of the fields $\partial_\mu\phi^I$ and $\partial_\mu\psi$, namely $Z^I \equiv \partial_\mu\phi^I\partial^\mu\psi$, which could be useful, together with the other invariants, to describe in a more natural way the dynamics of a fluid in some particular conditions, such as, for example, the *superfluid phase transition*. While the Z^I are not invariant under the chemical shift symmetry, a particular combination of the Z^I with the other Poincaré invariants is actually invariant under chemical-shift symmetry and its introduction also provides a natural understanding of the kinetic term of ψ. It actually coincides with $y^2 = u^\mu u^\nu \partial_\mu\psi\partial_\nu\psi$, u^μ being the fluid four-velocity.

In this Letter we want to substantiate the observation given in Ref. [1] by explaining why the introduction of the invariants Z^I can describe a superfluid as a spontaneous broken phase of a field theory invariant under chemical shift symmetry.

In the following, we first give a short review of the Lagrangian formalism for fluid mechanics as presented in [8,9], paying particular attention to the chemical shift symmetry and to the derivation of the energy–momentum tensor. Then we discuss the properties of the new Poincaré invariants Z^I, presented at the bosonic level in [1], further extending them to a supersymmetric setting. In the application to the superfluid we show how they provide a better

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[1] See also [11] for a complete review.
[2] Their precise definition will be specified in the final part of this note.

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parametrization of the superfluid phase separation.[3] The new invariants would be also useful for the quantum extension of the theoretical description of fluids along the lines of [14]. Finally we give the extension of the supersymmetric approach to the fluid dynamics given in [1] using the new variables Z^I.

The field-theory Lagrangian approach to fluid dynamics was developed in Refs. [6–9]. It is based on the use of the comoving coordinates of the fluid as fundamental fields. We will adopt the same notations as [8].

Working, for the sake of generality, in $d+1$ space–time dimensions, one introduces d scalar fields $\phi^I(x^I,t)$, $I=1,\dots,d$, as Lagrangian comoving coordinates of a fluid element at a point x^I and time t, such that a background is described by $\phi^I = x^I$ and requires, in the absence of gravitation, the following symmetries:

$$\delta\phi^I = a^I \quad \bigl(a^I = \text{const.}\bigr), \tag{1}$$

$$\phi^I \to O^I{}_J\phi^J \quad \bigl(O^I{}_J \in \mathrm{SO}(d)\bigr), \tag{2}$$

$$\phi^I \to \xi^I(\phi), \quad \det\bigl(\partial\xi^I/\partial\phi^J\bigr) = 1. \tag{3}$$

The following current respects the symmetries (1)–(3):

$$J^\mu = \frac{1}{d!}\epsilon^{\mu,\nu_1,\dots,\nu_d}\epsilon_{I_1,\dots,I_d}\partial_{\nu_1}\phi^{I_1}\dots\partial_{\nu_d}\phi^{I_d}, \tag{4}$$

and enjoys the important property that its projection along the comoving coordinates does not change:

$$J^\mu\partial_\mu\phi^I = 0. \tag{5}$$

This is equivalent to saying that the spatial d-form current $J^{(d)} = -\star^{d+1} J^{(1)}$, where

$$\begin{aligned} J^{(1)} &= \frac{1}{d!}\epsilon_{\mu\nu_1\dots\nu_d}\epsilon_{I_1\dots I_d}\partial^{\nu_1}\phi^{I_1}\dots\partial^{\nu_d}\phi^{I_d}\,dx^\mu \\ &= (-1)^d\star^{d+1}\left(\frac{1}{d!}\epsilon_{I_1\dots I_d}\,d\phi^{I_1}\wedge\dots\wedge d\phi^{I_d}\right), \end{aligned} \tag{6}$$

is closed identically, that is it is an exact form. Hence it is natural to define the fluid four-velocity as aligned with J^μ:

$$J^\mu = bu^\mu \to b = \sqrt{-J^\mu J_\mu} = \sqrt{\det\bigl(B^{IJ}\bigr)}, \tag{7}$$

where $B^{IJ} \equiv \partial_\mu\phi^I\partial^\mu\phi^J$. From a physical point of view, the property of J^μ to be identically closed identifies it with the entropy current of the perfect fluid in the absence of dissipative effects, so that $b = s$, s being the entropy density.

If there is a conserved charge (particle number, electric charge etc.), one introduces a new field $\psi(x^I,t)$ which is a phase, that is it transforms under U(1) as follows

$$\psi \to \psi + c \quad (c = \text{const.}). \tag{8}$$

Moreover, if the charge flows with the fluid, charge conservation is obeyed separately by each volume element. This means that the charge conservation is not affected by an arbitrary comoving position-dependent transformation

$$\psi \to \psi + f\bigl(\phi^I\bigr), \tag{9}$$

f being an arbitrary function. This extra symmetry requirement on the Lagrangian is dubbed *chemical-shift symmetry*. Using the entropy current J^μ one finds that, by virtue of Eq. (5), the quantity $J^\mu\partial_\mu\psi$ is invariant under (9).

From these premises the authors of [8] constructed the low energy Lagrangian respecting the above symmetries. To lowest order in a derivative expansion the Lagrangian will depend on the first derivatives of the fields through invariants respecting the symmetries (1)–(3), (8):

$$\mathcal{L} = \mathcal{L}\bigl(\partial\phi^I, \partial\psi\bigr). \tag{10}$$

In principle, there are two such invariants constructed out of J^μ and $\partial_\mu\psi$, namely b and $J^\mu\partial_\mu\psi$, so that the (Poincaré invariant) action functional can be written as follows:

$$S = \int d^{d+1}x\,F(b,y), \tag{11}$$

where y is

$$y = u^\mu\partial_\mu\psi = \frac{J^\mu\partial_\mu\psi}{b}. \tag{12}$$

By coupling (11) to a gravity background, one can obtain the energy–momentum tensor by taking, as usual, the variation of S with respect to a background (inverse) metric $g^{\mu\nu}$:

$$T_{\mu\nu} = (yF_y - bF_b)u_\mu u_\nu + \eta_{\mu\nu}(F - bF_b). \tag{13}$$

On the other hand, from classical fluid dynamics, we also have

$$T_{\mu\nu} = (p+\rho)u_\mu u_\nu + \eta_{\mu\nu}p, \tag{14}$$

from which we identify the pressure and density

$$\rho = yF_y - F \equiv yn - F, \qquad p = F - bF_b. \tag{15}$$

Comparing the two expressions of the energy–momentum tensor one can derive the relations between the thermodynamical functions and the field-theoretical quantities (see [8] for a complete review). In particular, it turns out that the quantity y defined in Eq. (12) coincides with the chemical potential μ. We conclude that the Lagrangian density of a perfect fluid is a function of s and μ

$$F = F(s,\mu). \tag{16}$$

The results presented here can be straightforwardly generalized to the supersymmetric case, where the comoving coordinates ϕ^I and phase ψ are extended to superfields. This was given in [1].

The Lagrangian (16) used above depends on the two Poincaré-invariant variables b and y, but at first sight it does not seem to allow for the presence of a kinetic term for the dynamical field ψ, namely $X = \partial_\mu\psi\partial^\mu\psi$. This term respects the translational invariance (8), and was actually considered in [9], but fails to satisfy the *chemical-shift symmetry* (9). However, as we are going to see, the kinetic term for ψ is in fact included in y^2.

To show this, let us observe that the Poincaré invariants one can build from the fields $\partial_\mu\phi^I$ and $\partial_\mu\psi$ are given by B^{IJ}, y, X together with the variables $Z^I \equiv \partial_\mu\psi\partial^\mu\phi^I$. Under chemical-shift symmetry they transform as

$$\begin{aligned} &\delta B^{IJ} = 0, \\ &\delta X = 2\partial_I f Z^I, \\ &\delta y = 0, \\ &\delta Z^I = \partial_J f B^{IJ}. \end{aligned} \tag{17}$$

We note that B^{IJ} and y are invariant under the chemical-shift symmetry, while the other two quantities X and Z^I are not. However we can construct out of them a new invariant $I(B,X,Z)$:

$$I(B,X,Z) = X - Z^I Z^J\bigl(B^{-1}\bigr)_{IJ}, \tag{18}$$

[3] See [12,13] for a review on superfluids in the context of holography and fluid/gravity correspondence.

which is inert under chemical shift symmetry as well. In fact the new invariant (18) actually coincides with $-y^2$, since

$$I(B,X,Z) = \partial_\mu\psi\partial_\nu\psi\left(\eta^{\mu\nu} - \partial^\mu\phi^I\partial^\nu\phi^J B^{-1}_{IJ}\right) = \partial_\mu\psi\partial_\nu\psi\left(-u^\mu u^\nu\right) = -y^2 \tag{19}$$

by virtue of the identity (see for example [8,9])

$$\partial_\mu\phi^I\partial_\nu\phi^J B^{-1}_{IJ} = \eta_{\mu\nu} + u_\mu u_\nu. \tag{20}$$

In terms of X, Z, B a Poincaré invariant Lagrangian, enjoying the symmetries (1)–(3) and (8), and generalizing (16), can be constructed. It should be an $\mathrm{SO}(d)$-invariant functional of the fields B^{IJ}, X, Z^I:

$$S = \int \mathcal{F}(B,X,Z)\, d^{d+1}x. \tag{21}$$

As we shall see in the sequel, a Lagrangian of this kind is useful to describe physical situations where the chemical-shift symmetry is spontaneously broken, and a particular instance of it was considered, for example, in [9] to describe a superfluid at $T=0$.

The restricted form $F(b,y)$ is however required in all the cases where the chemical-shift invariance of the Lagrangian is expected. In this case, the fields X, Z^I can only appear in the invariant combination (18), and one recovers the standard form of the Lagrangian $F(b,y)$. We will discuss a possible application of this extended formalism below, when we consider the case of the superfluid phase transition, where the chemical-shift symmetry is spontaneously broken, together with its constant part (8).

Let us develop the dynamics deriving from the action principle (21). The variation of the Poincaré-invariant fields under a generic variation of ϕ^I and of ψ are

$$\delta B^{IJ} = -2\, d\phi^I \wedge \star d\delta\phi^J,$$
$$\delta X = -2\, d\psi \wedge \star d\delta\psi,$$
$$\delta Z^I = -d\delta\psi \wedge \star d\phi^I - d\psi \wedge \star d\delta\phi^I.$$

The variation of y can be given in terms of the variation of B^{IJ}, X, Z^I. The following equations of motion are obtained:

$$d\star J^{(1)}_I = d\left[2\mathcal{F}_{B^{IJ}}\star d\phi^J + \mathcal{F}_{Z^I}\star d\psi\right] = 0, \tag{22}$$
$$d\star j^{(1)} = d\left[2\mathcal{F}_X\star d\psi + \mathcal{F}_{Z^I}\star d\phi^I\right] = 0, \tag{23}$$

where $\mathcal{F}_{B^{IJ}} = \partial\mathcal{F}/\partial B^{IJ}$, $\mathcal{F}_X = \partial\mathcal{F}/\partial X$, $\mathcal{F}_{Z^I} = \partial\mathcal{F}/\partial Z^I$ and we have introduced the two currents:

$$J^{(1)}_I = 2\mathcal{F}_{B^{IJ}}\, d\phi^J + \mathcal{F}_{Z^I}\, d\psi, \tag{24}$$
$$j^{(1)} = 2\mathcal{F}_X\, d\psi + \mathcal{F}_{Z^I}\, d\phi^I. \tag{25}$$

It is straightforward to verify that the above quantities are the Noether currents associated with the constant translational symmetries $\phi^I \to \phi^I + c^I$ and $\psi \to \psi + c$.

It is interesting to write down the energy momentum tensor for the generalized Lagrangian $\mathcal{F}[B,X,Z]$. The new energy–momentum tensor $\tilde{T}_{\mu\nu}$ is now

$$\tilde{T}_{\mu\nu} = \eta_{\mu\nu}\mathcal{F} - 2\left(\frac{\partial\mathcal{F}}{\partial X}\Omega_\mu\Omega_\nu + \frac{\partial\mathcal{F}}{\partial Z^I}\Omega_\mu\Pi^I_\nu + \frac{\partial\mathcal{F}}{\partial B^{IJ}}\Pi^I_\mu\Pi^J_\nu\right). \tag{26}$$

To recover a Lagrangian enjoying invariance under chemical-shift symmetry

$$\mathcal{F}[B,X,Z] = F[s,y], \tag{27}$$

one has to consider the case where the fields B^{IJ}, X and Z^I only appear in the invariant combinations (19) and $b^2 = \det(B)$. Consequently, the corresponding energy–momentum tensor is retrieved from (26) by using the relations $y = y(B,X,Z)$ and $s = b = \sqrt{\det(B^{IJ})}$.

As already emphasized, the generalized Lagrangian $\mathcal{F}$ can be useful for describing fluid dynamics. One possible application can be found in the description of the 2-fluid model for superfluidity, in the same spirit as the approach in [9]. Let us recall few well known properties of the Helium superfluid phase transition:

- Above the critical temperature T_C the fluid has a normal behavior and is invariant under the chemical-shift symmetry [8]. It is described in terms of the comoving coordinates $\phi^I(x)$ and by the $U(1)$-phase field $\psi(x)$.
  On the other hand, below the critical temperature the chemical-shift symmetry is spontaneously broken, giving rise to the superfluid. In particular, at $T=0$, the superfluid is completely described in terms of ψ.
- One can consider, following [9], an isotropic and homogeneous background where

$$\psi = y_0 t, \qquad \phi^I = b_0^{1/3} x^I. \tag{28}$$

  It corresponds to taking a configuration where the fields ϕ^I are comoving with the normal fluid part (which is at rest in this background), the superfluid field ψ being in relative motion with respect to it. Note that the loss of interactions between the two fluids is expressed by the property that $Z^I = \partial_\mu\psi\partial^\mu\phi^I = 0$ in the background.
- Small perturbations about the background (28):

$$\psi = y_0\left(t + \pi^0(x)\right), \qquad \phi^I = b_0^{1/3}\left(x^I + \pi^I(x)\right) \tag{29}$$

  introduce a small interaction term $Z^I \neq 0$. Note that the quantity $B^{-1}_{IJ} Z^I Z^J = \epsilon$ stays small in this regime, even if $\phi^I \to 0$ as $T \to 0$.

Given these considerations, we can make use of the relation (18) to observe that at very low temperatures the quantity

$$y^2 = -X + B^{-1}_{IJ} Z^I Z^J = -X + \epsilon \tag{30}$$

approaches the value $-X$, which is not invariant under the chemical-shift symmetry. In this regime the Lagrangian $F(b,y)$ can be expanded in powers of ϵ around the background, neglecting contributions $\mathcal{O}(\epsilon^k)$, for a finite k. The effective Lagrangian at low temperatures, which is no longer invariant under chemical-shift symmetry, becomes then of the kind $\mathcal{F}[X, Z^I, B^{IJ}]$, and reduces to $\mathcal{F}[X]$ for $T \to 0$.

Let us finally observe that above the critical temperature the theory is still described by the Poincaré invariant fields X, B^{IJ}, Z^I including the kinetic terms of ϕ^I and ψ which, however, only appear in the chemical-shift invariant combinations b, y since Z^I is not small anymore, so that the form $F(s,y)$ of the Lagrangian is recovered at $T > T_C$. In the quantum regime, as in [14], the expansion around the background by using the new invariant might lead to a natural reorganization of the perturbation theory.

In the rest of this Letter we give the supersymmetric extension of this model, following the approach developed in [1]. We use the same formalism and notations as given in Ref. [1]. A basis of 1-superforms in a general *rigid* $(d+1|m)$-superspace can be given in terms of the supervielbein $\{\Pi^a, \Psi^\alpha\}$ by:

$$\Pi^a = d\phi^a + \frac{i}{2}\bar{\theta}\Gamma^a\, d\theta, \qquad \Psi^\alpha = d\theta^\alpha, \tag{31}$$

where $a = (0, I)$ $(I = 1, \dots d)$ and $\alpha = (1, \dots m)$ run over the bosonic and fermionic directions of superspace respectively. Here Γ^a are the Clifford algebra Γ-matrices in $(d+1)$-dimensions, while θ, and $d\theta$ denote the matrix form of the *Majorana* spinors in the $m = 2^{[(d+1)/2]}$-dimensional spinor representation of $SO(d,1)$.[4] In particular $\bar\theta \equiv \theta^\dagger \Gamma^0 = \theta^T C$, $C = (C_{\alpha\beta})$ being the charge-conjugation matrix. The space-like fields $\phi^I(x,t)$ can be taken as the comoving coordinate fields of the bosonic theory, while the spinors $\theta^\alpha(x,t)$ are the fermionic coordinates of superspace and we added a time-like bosonic field ϕ^0 to complete the superspace (see also [11]).

Together with the supersymmetric extension of $d\phi^I$ we also introduce a 1-form Ω, representing the supersymmetric extension of the chemical shift field-strength $d\psi$:

$$\Omega = d\psi + i\bar\tau\, d\theta, \tag{32}$$

τ being a new Majorana spinor.

The supersymmetry transformations of the fundamental field ϕ^a and of the supervielbein Π^a (supersymmetric extension of $\partial_\mu \phi^I$) are

$$\delta_\epsilon \phi^a = \frac{i}{2}\bar\epsilon \Gamma^a \theta, \qquad \delta_\epsilon \theta = \epsilon, \tag{33}$$

$$\delta_\epsilon \Pi^a = i\bar\epsilon \Gamma^a\, d\theta, \qquad \delta_\epsilon \psi = d\epsilon = 0, \tag{34}$$

where ϵ is the constant spinorial parameter of supersymmetry.

One further assumes that the phase $\psi(x)$ and its supersymmetric partner $\tau(x)$ are invariant under the rigid supersymmetry generated by the Killing vector of supersymmetry $\vec\epsilon$. We may, however, extend the chemical shift "internal" symmetry to superspace performing the following superdiffeomorphism on ψ:

$$\delta\psi = f(\phi, \theta) \tag{35}$$

with $f(\phi,\theta)$ arbitrary superfield. Assuming $\delta\tau_\alpha = D_\alpha f(x,\theta)$, where

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + \frac{i}{2}\bar\theta^\beta \Gamma^I_{\beta\alpha}\frac{\partial}{\partial\phi^I}$$

the 1-form Ω acquires the following chemical shift transformation:

$$\delta\Omega = \frac{\partial f}{\partial\phi^I}\Pi^I. \tag{36}$$

In terms of the above variables, we can build the following quantities that generalize the corresponding bosonic Poincaré-invariant fields:

$$\begin{aligned}
B^{IJ} &= -\Pi^I \wedge \star \Pi^J = \Pi^I_\mu \Pi^{J\mu}\, d^{d+1}x = \hat B^{IJ}\, d^{d+1}x,\\
X &= -\Omega \wedge \star\Omega = \Omega_\mu g^{\mu\nu}\Omega_\nu d^{d+1}x = \hat X\, d^{d+1}x,\\
Y &= \Omega \wedge \Pi^1 \wedge \cdots \wedge \Pi^d\\
&= \frac{1}{d!}\epsilon^{\mu\nu_1\dots\nu_d}\epsilon_{I_1\dots I_d}\Omega_\mu \Pi^{I_1}_{\nu_1}\dots\Pi^{I_d}_{\nu_d}\, d^{d+1}x = \hat Y\, d^{d+1}x,\\
Z^I &= -\Omega\wedge\star\Pi^I = \Omega_\mu g^{\mu\nu}\Pi^I_\nu\, d^{d+1}x = \hat Z^I\, d^{d+1}x.
\end{aligned} \tag{37}$$

We denote by the same letter, though with a hat on the top, the corresponding factor multiplying the volume form $d^{d+1}x$, for instance $\hat B^{IJ} = \Pi^I_\mu g^{\mu\nu}\Pi^J_\nu$. Note in particular that [8]:

$$\hat b = \sqrt{\det \hat B^{IJ}}. \tag{38}$$

[4] For *Majorana–Weyl* spinors, the dimension of the representation is instead $m = 2^{[d/2]}$.

The Poincarè invariant quantities defined in (37) under chemical-shift symmetry transform as follows

$$\begin{aligned}
&\delta B^{IJ} = 0,\\
&\delta X = -2\Omega\wedge\star\delta\Omega = -2\partial_I f\,\Omega\wedge\star\Pi^I = 2\partial_I f Z^I,\\
&\delta Y = 0,\\
&\delta Z^I = -\delta\Omega\wedge\star\Pi^I = -\partial_J f\,\Pi^J\wedge\star\Pi^I = \partial_J f B^{IJ},
\end{aligned} \tag{39}$$

so that B^{IJ} and Y are invariant under the chemical shift symmetry, while the other two quantities X and Z^I are not. The supersymmetric generalization of the invariant (19) takes now the form

$$I(\hat B, \hat X, \hat Z) = \hat X - \hat Z^I \hat Z^J \big(\hat B^{-1}\big)_{IJ}. \tag{40}$$

The new invariant (18) actually coincides with $-\hat Y^2/\hat b^2$. This was illustrated at the bosonic level in Eq. (19). To show that the bosonic result extends straightforwardly to the supersymmetric case we need the relation

$$\Pi^I_\mu \hat u^\mu = 0, \tag{41}$$

in terms of the "supervelocity" superfield $\hat u^\mu$ (generalizing (7) to a supersymmetric setting):

$$\hat u^\mu = (-)^{d+1}\hat b^{-1}\epsilon^{\mu\mu_1\dots\mu_d}\Pi^1_{\mu_1}\dots\Pi^d_{\mu_d}. \tag{42}$$

The Lagrangian describing the dynamics of the supersymmetric fluid should then be an $SO(d)$-invariant functional of the fields B^{IJ}, X, Z^I:

$$S = \int \mathcal{F}(\hat B, \hat X, \hat Z)\, d^{d+1}x, \tag{43}$$

which is an obvious generalization of the bosonic case.

The variations of the superfields B^{IJ}, X, Z^I under a generic variation of the bosonic fields ϕ^I and ψ are

$$\begin{aligned}
&\delta B^{IJ} = -2\Pi^I\wedge\star d\delta\phi^J,\\
&\delta X = -2\Omega\wedge\star d\delta\psi,\\
&\delta Z^I = -d\delta\psi\wedge\star\Pi^I - \Omega\wedge\star d\delta\phi^I,
\end{aligned} \tag{44}$$

and the following equations of motion are obtained:

$$d\star J^{(1)}_I = d\big[2\mathcal{F}_{B^{IJ}}\star\Pi^J + \mathcal{F}_{Z^I}\star\Omega\big] = 0, \tag{45}$$

$$d\star j^{(1)} = d\big[2\mathcal{F}_X\star\Omega + \mathcal{F}_{Z^I}\star\Pi^I\big] = 0, \tag{46}$$

where $\mathcal{F}_{B^{IJ}} = \partial\mathcal{F}/\partial\hat B^{IJ}$, $\mathcal{F}_X = \partial\mathcal{F}/\partial\hat X$, $\mathcal{F}_{Z^I} = \partial\mathcal{F}/\partial\hat Z^I$ and we have introduced the two currents:

$$J^{(1)}_I = 2\mathcal{F}_{B^{IJ}}\Pi^J + \mathcal{F}_{Z^I}\Omega, \tag{47}$$

$$j^{(1)} = 2\mathcal{F}_X\Omega + \mathcal{F}_{Z^I}\Pi^I. \tag{48}$$

Furthermore, a general variation of the fermionic fields θ and τ, gives

$$\begin{aligned}
&\delta B^{IJ} = -i\Pi^I\wedge\star\big(\delta\bar\theta\Gamma^J d\theta + \bar\theta\Gamma^J d\delta\theta\big),\\
&\delta X = 2i\Omega\wedge\star(\delta\bar\tau\, d\theta + \bar\tau\, d\delta\theta),\\
&\delta Z^I = i(\delta\bar\tau\, d\theta + \bar\tau\, d\delta\theta)\wedge\star\Pi^I + \frac{i}{2}\Omega\wedge\star\big(\delta\bar\theta\Gamma^I d\theta + \bar\theta\Gamma^I d\delta\theta\big),
\end{aligned} \tag{49}$$

so that the corresponding *fermionic* equations of motion are

$$J^\mu_I \Gamma^I \partial_\mu\theta + \eta_C j^\mu \partial_\mu\tau = 0, \tag{50}$$

where η_C is the sign appearing in the relation $\bar{\tau}\, d\theta = \eta_C\, d\bar{\theta}\, \tau$ and depends on the property of the charge conjugation matrix in $(d+1)$-dimensions.

Finally we perform a supersymmetry transformations of the fields B^{IJ}, X, Z^I using Eqs. (33) and (34) (note that the variable X is invariant under supersymmetry). We obtain

$$\delta_\epsilon S = (-)^{d+1} \int {}^*\big(J_I^{(1)}\big) \wedge i\bar{\epsilon}\Gamma^I\, d\theta, \tag{51}$$

so that by partial integration we find

$$\delta_\epsilon S = -\int \big(d^* J_I^{(1)}\big) i\bar{\epsilon}\Gamma^I \theta \tag{52}$$

and by virtue of the equation of motion (45) we conclude

$$\delta_\epsilon S = 0, \tag{53}$$

that is *the action is invariant under supersymmetry*.

The link between the above relations and the field equations given in [1] is easily retrieved by recalling that $\hat{Y} = \hat{Y}(\hat{X}, \hat{Z}, \hat{B})$ through the relation (40) and $\hat{b} = \sqrt{\det(\hat{B}^{IJ})}$.

For completeness, we also give the general expression of the supercurrent:

$$j_S = J_I \Gamma^I \theta + \eta_C j \tau. \tag{54}$$

The present formalism can provide a framework for studying supergravity/supersymmetric fluid correspondence along the lines of [2]. In particular, it might provide the natural ground for studying the dissipative effects as in [15–17] and a systematically computation of entropy current corrections [18–20].

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References

[1] L. Andrianopoli, R. D'Auria, P.A. Grassi, M. Trigiante, Entropy current formalism for supersymmetric theories, arXiv:1304.2206 [hep-th].
[2] S. Bhattacharyya, V.E. Hubeny, S. Minwalla, M. Rangamani, Nonlinear fluid dynamics from gravity, J. High Energy Phys. 0802 (2008) 045, arXiv:0712.2456 [hep-th].
[3] M. Rangamani, Gravity and hydrodynamics: lectures on the fluid-gravity correspondence, Class. Quantum Gravity 26 (2009) 224003, arXiv:0905.4352 [hep-th].
[4] V.E. Hubeny, S. Minwalla, M. Rangamani, The fluid/gravity correspondence, arXiv:1107.5780 [hep-th].
[5] J. Bhattacharya, S. Bhattacharyya, M. Rangamani, Non-dissipative hydrodynamics: effective actions versus entropy current, arXiv:1211.1020 [hep-th].
[6] J.D. Brown, Action functionals for relativistic perfect fluids, Class. Quantum Gravity 10 (1993) 1579, arXiv:gr-qc/9304026.
[7] S. Dubovsky, T. Gregoire, A. Nicolis, R. Rattazzi, Null energy condition and superluminal propagation, J. High Energy Phys. 0603 (2006) 025, arXiv:hep-th/0512260.
[8] S. Dubovsky, L. Hui, A. Nicolis, D.T. Son, Effective field theory for hydrodynamics: thermodynamics, and the derivative expansion, Phys. Rev. D 85 (2012) 085029, arXiv:1107.0731 [hep-th].
[9] A. Nicolis, Low-energy effective field theory for finite-temperature relativistic superfluids, arXiv:1108.2513 [hep-th].
[10] S. Dubovsky, L. Hui, A. Nicolis, Effective field theory for hydrodynamics: Wess–Zumino term and anomalies in two spacetime dimensions, arXiv:1107.0732 [hep-th].
[11] R. Jackiw, V.P. Nair, S.Y. Pi, A.P. Polychronakos, Perfect fluid theory and its extensions, J. Phys. A 37 (2004) R327, arXiv:hep-ph/0407101.
[12] C.P. Herzog, N. Lisker, P. Surowka, A. Yarom, Transport in holographic superfluids, J. High Energy Phys. 1108 (2011) 052, arXiv:1101.3330 [hep-th].
[13] J. Bhattacharya, S. Bhattacharyya, S. Minwalla, Dissipative superfluid dynamics from gravity, J. High Energy Phys. 1104 (2011) 125, arXiv:1101.3332 [hep-th].
[14] S. Endlich, A. Nicolis, R. Rattazzi, J. Wang, The quantum mechanics of perfect fluids, J. High Energy Phys. 1104 (2011) 102, arXiv:1011.6396 [hep-th].
[15] K. Kontoudi, G. Policastro, Diffusion constant of supercharge density in $N = 4$ SYM at finite chemical potential, J. High Energy Phys. 1211 (2012) 044, arXiv:1206.2067 [hep-th].
[16] C. Hoyos, B. Keren-Zur, Y. Oz, Supersymmetric sound in fluids, J. High Energy Phys. 1211 (2012) 152, arXiv:1206.2958 [hep-ph].
[17] J. Erdmenger, S. Steinfurt, A universal fermionic analogue of the shear viscosity, arXiv:1302.1869 [hep-th].
[18] S. Bhattacharyya, V.E. Hubeny, R. Loganayagam, G. Mandal, S. Minwalla, T. Morita, M. Rangamani, H.S. Reall, Local fluid dynamical entropy from gravity, J. High Energy Phys. 0806 (2008) 055, arXiv:0803.2526 [hep-th].
[19] R. Loganayagam, Entropy current in conformal hydrodynamics, J. High Energy Phys. 0805 (2008) 087, arXiv:0801.3701 [hep-th].
[20] G. Plewa, M. Spalinski, Entropy currents from holography in hydrodynamics with charge, arXiv:1304.6705 [hep-th].